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Martin Raussen and Krzysztof Ziemiański

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Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 99 40 ▪ Telefax: +45 99 40 35 48
URL: http://www.math.aau.dk

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HOMOLOGY OF SPACES OF DIRECTED PATHS ON EUCLIDEAN CUBICAL COMPLEXES

MARTIN RAUSSEN AND KRZYSZTOF ZIEMIAŃSKI

Abstract. We compute the homology of the spaces of directed paths on a certain class of cubical subcomplexes of the directed Euclidean space $\mathbb{R}^n$ by a recursive process. We apply this result to calculate the homology and cohomology of the space of directed loops on the $(n-1)$-skeleton of the directed torus $\vec{T}^n$.

1. Introduction

One of the most important problems of Directed Algebraic Topology is calculating the homotopy type of spaces of directed paths $\vec{P}(X)_x^y$ between two points x, y of a directed space X. This problem seems to be very difficult in general case; however several results were recently obtained. The first author gave in a series of papers [10, 11, 12, 13] a description of the homotopy type $\vec{P}(X)_x^y$ in the case when X is a directed cube from which collections of homothetic rectangular areas are removed. In this paper, we present explicit calculations of the homology and cohomology of directed path spaces in important particular cases.

1.1. **d-spaces.** A d-space (Grandis[6, 7]) is a pair $(X, \vec{P}(X))$, where X is a topological space, and $\vec{P}(X)$ is a family of paths on X that contains all constant paths and that is closed under non-decreasing reparametrizations and concatenations. The family $\vec{P}(X)$ is called *a d-structure* on X, and paths which belong to $\vec{P}(X)$ will be called *directed paths* or d-paths. For $x, y \in X$ define *the directed path space* from x to y as

$$\vec{P}(X)_x^y = \{\alpha \in \vec{P}(X) : \ \alpha(0) = x \ \wedge \ \alpha(1) = y\}. \tag{1.1}$$

The *directed real line* $\vec{\mathbb{R}}$ is the d-space with underlying space $\mathbb{R}$ and $\vec{P}(\vec{\mathbb{R}})$ the set of all *non-decreasing* paths. *Directed Euclidean space* $\vec{\mathbb{R}}^n$ is the product $\vec{\mathbb{R}} \times \cdots \times \vec{\mathbb{R}}$ with the product d-structure $\vec{P}(\vec{\mathbb{R}}^n) = \vec{P}(\vec{\mathbb{R}}) \times \cdots \times \vec{P}(\vec{\mathbb{R}})$. Finally, the *directed torus* $\vec{T}^n$ is the quotient $\vec{\mathbb{R}}^n/\mathbb{Z}^n$; a path on $\vec{T}^n$ is directed iff it lifts to a directed path on $\vec{\mathbb{R}}^n$.

1.2. **Euclidean cubical complexes.** *An elementary cube* in $\mathbb{R}^n$ is a product

$$[k_1, k_1 + e_1] \times \cdots \times [k_n, k_n + e_n] \subseteq \mathbb{R}^n,$$

where $k_i \in \mathbb{Z}$ and $e_i \in \{0, 1\}$; the dimension of a cube is the sum $\sum_1^n e_i$. A *Euclidean cubical complex* is defined to be a subset $K \subseteq \mathbb{R}^n$ that is a union of elementary cubes. The *d-skeleton of* K, denoted by $K_{(d)}$, is the union of all elementary cubes contained in K which have dimensions less than or equal to d. Euclidean space can be identified with the geometric realization of a suitable pre-cubical set such that realizations of cubes of this pre-cubical set

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are elementary cubes in $\mathbb{R}^n$, and Euclidean cubical sets are the geometric realizations of pre-cubical subsets of that pre-cubical set. Every cubical complex is provided with the directed structure inherited from $\vec{\mathbb{R}}^n$.

Euclidan cubical complexes are special cases of general cubical complexes which are geometric realizations of general pre-cubical sets; cf Brown and Higgins [1].

1.3. **Notation.** Points on $\mathbb{R}^n$ will be denoted by bold letters, their coordinates by regular ones with suitable indices; for example $\mathbf{a} = (a_1, \dots, a_n)$. Furthermore, we will write $\mathbf{0}$ for $(0, \dots, 0)$ and $\mathbf{1}$ for $(1, \dots, 1)$. Three kinds of comparison operators between points of $\mathbb{R}^n$ will be used:

$$\begin{aligned} \mathbf{a} \le \mathbf{b} &\Leftrightarrow \forall_{i=1}^n \; a_i \le b_i \\ \mathbf{a} < \mathbf{b} &\Leftrightarrow \forall_{i=1}^n \; a_i \le b_i \; \wedge \; \mathbf{a} \neq \mathbf{b} \\ \mathbf{a} \ll \mathbf{b} &\Leftrightarrow \forall_{i=1}^n \; a_i < b_i. \end{aligned}$$

In analogy to the one-dimensional case, write $[\mathbf{a}, \mathbf{b}] := \{\mathbf{t} : \; \mathbf{a} \le \mathbf{t} \le \mathbf{b}\}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Finally, let $|\mathbf{x}| := \sum_{i=1}^n |x_i|$ for $\mathbf{x} \in \mathbb{R}^n$ denote the l_1-norm; the l_1-metric $\mu(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ on $\mathbb{R}^n$ is compatible with standard Euclidean topology. Notice that $|\mathbf{x} - \mathbf{y}| = ||\mathbf{x}| - |\mathbf{y}||$ whenever $\mathbf{x} \le \mathbf{y}$.

1.4. **The main theorem.** Let $\mathbf{k} \in \mathbb{Z}^n$, $n \ge 3$, and let $K \subseteq [\mathbf{0}, \mathbf{k}] \subseteq \mathbb{R}^n$ denote a Euclidean cubical complex that *contains* the $(n-1)$-skeleton $[\mathbf{0}, \mathbf{k}]_{(n-1)} \subseteq K$. A *cube sequence in K of length r* is a sequence

$$[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \dots \ll \mathbf{a}^r \le \mathbf{k}], \tag{1.2}$$

where $\mathbf{a}^i \in \mathbb{Z}^n$ and such that $[\mathbf{a}^i - \mathbf{1}, \mathbf{a}^i] \not\subseteq K$. Let $CS_r(K)$ be the set of cube sequences of length r and define the graded abelian group $A_*(K)$ by

$$A_m(K) = \begin{cases} \mathbb{Z}[CS_{m/(n-2)}(K)] & \text{if } n-2 \text{ divides } m \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The main theorem of this paper is the following

Theorem 1.4. *The graded abelian groups $H_*(\vec{P}(K))_{\mathbf{0}}^{\mathbf{k}}$ and $A_*(K)$ are isomorphic.*

1.5. **An application: The space of paths on the $(n-1)$-skeleton of the directed torus $\vec{T}^n$.** Assume that $n \ge 3$ and $d \ge 2$. Let $\vec{T}^n_{(d)} := \vec{\mathbb{R}}^n_{(d)}/\mathbb{Z}^n$ be the d-skeleton of $\vec{T}^n$. Every directed path $\alpha \in \vec{P}(\vec{T}^n_{(d)})_{\mathbf{0}}^{\mathbf{0}}$ represents a class $\mathbf{k} \in \pi_1(\vec{T}^n_{(d)})_{\mathbf{0}} \cong \mathbb{Z}^n$. By passing to the universal covering of $T^n_{(d)}$ we see that α lifts uniquely to a path $\tilde{\alpha} \in \vec{P}(\mathbb{R}^n_{(d)})_{\mathbf{0}}^{\mathbf{k}}$. Since $\tilde{\alpha}$ is directed, the class $\mathbf{k}$ is non-negative: $\mathbf{0} \le \mathbf{k}$. Since also directed homotopies lift uniquely, we obtain a homeomorphism

$$\vec{P}(T^n_{(d)})_{\mathbf{0}}^{\mathbf{0}} \cong \coprod_{\mathbf{0} \le \mathbf{k} \in \mathbb{Z}^n} \vec{P}(\mathbb{R}^n_{(d)})_{\mathbf{0}}^{\mathbf{k}}. \tag{1.5}$$

If $d = n-1$, we can apply the main theorem to obtain an isomorphism

$$H_*(\vec{P}(T^n_{(n-1)})_{\mathbf{0}}^{\mathbf{0}}) \cong \bigoplus_{\mathbf{0} \le \mathbf{k} \in \mathbb{Z}^n} A_*(\mathbb{R}^n_{(n-1)} \cap [\mathbf{0}, \mathbf{k}]). \tag{1.6}$$

The following proposition allows to calculate the Betti numbers of the components:

Proposition 1.7. *For* $\mathbf{k} \geq \mathbf{0}$ *we have*

$$\dim H_{r(n-2)}(\vec{P}(\mathbb{R}^n_{(n-1)})^{\mathbf{k}}_{\mathbf{0}}) = \dim A_{r(n-2)}(\mathbb{R}^n_{(n-1)} \cap [\mathbf{0}, \mathbf{k}]) = \binom{k_1}{r}\binom{k_2}{r}\cdots\binom{k_n}{r}.$$

Proof. The map

$$CS_r(\mathbb{R}^n_{(n-1)} \cap [\mathbf{0}, \mathbf{k}]) \ni [\mathbf{a}^1 \ll \cdots \ll \mathbf{a}^r \leq \mathbf{k}] \mapsto \\ (\{a^1_1, \ldots, a^r_1\}, \ldots, \{a^1_n, \ldots, a^r_n\}) \in 2^{\{1,\ldots,k_1\}} \times \cdots \times 2^{\{1,\ldots,k_n\}}$$

is clearly a bijection. The conclusion follows. □

Remark. An attempt to calculate the homology of $\vec{P}(\mathbb{R}^3_2)^{(k,l,m)}_{\mathbf{0}}$ using the poset description for the cell complex of the prod-simplicial complex homotopy equivalent to that path space according to Raussen [11] by "brute force" – even using sophisticated homology software – failed already for $k = l = m = 3$. The prod-simplicial complex in this case has dimension $klm(n-2)$; its homological dimension is only $\min\{k, l, m\}(n-2)$. This contrast was one of the motivations for looking for better descriptions of path spaces.

1.6. **Euclidean cubical complexes and concurrency.** One of the motivations for developing Directed Algebraic Topology goes back to particular models in concurrency theory, the so-called Higher Dimensional Automata, cf eg Pratt[8], van Glabbeek[15]. A particular class of Higher Dimensional Automata arises from semaphore or mutex models: Each processor records on a time line when it accesses (P) and relinquishes (V) a number of shared objects; the forbidden region F associated to such a PV-program (cf Dijkstra [3]) consists of a union of isothetic hyperrectangles $R^i \subset \vec{I}^n$ within an n-cube $\vec{I}^n \subset \vec{\mathbb{R}}^n$; cf Fajstrup etal. [5].

The particular Euclidean complexes whose path spaces we study correspond to PV-programs with the following two particular properties:

- All shared objects have arity $n-1$, ie, $n-1$ out of n but not all n processes can access the object at any given time;
- The PV-program for every indiviudal processor is of type $(PV)(PV)\ldots(PV)$ – a variety of shared objects is allowed. In particular, every access to a shared object is terminated before a new one is accessed. This has the consequence that the hyperretangles R^i and their projections to the axes do not overlap with each other.

No doubt that this represents a very particular case. On the other hand, our result seems to be the first non-trivial calculation of the homology of spaces of directed paths in closed form. Note that a description of a simplicial complex homotopy equivalent to directed paths in a torus with holes was obtained in Fajstrup [4].

The application in Section 1.5 – which motivated this line of investigation – shows that it is also possible to consider programs with loops. The case considered here corresponds to n looped processors of type $(PaVa)^* \parallel \cdots \parallel (PaVa)^*$.

1.7. **The case $n = 2$.** Only for $n = 2$, the path spaces are, in general, not connected – and therefore the result of a distributed programme may depend on the schedule. The method described above still works, but there is a slight twist due to the fact that cube sequences regardless of their length all contribute (only) to dimension 0:

The space $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ is a space consisting of contractible components. The number of components is $\beta_0(\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}) = |CS(K)| + 1$, the number of *all* cube sequences in K augmented by one. The reason is that both sides in the equation above obey to the recursion formula $a_{k+1,l+1} = \begin{cases} a_{k+1,l} + a_{k,l+1} - a_{kl} & [(k,l),(k+1,l+1)] \subset K \\ a_{k+1,l} + a_{k,l+1} & [(k,l),(k+1,l+1)] \not\subset K \end{cases}$ with start values $a_{k,0} = a_{0,l} = 1$.

In the particular case dealt with in Section 1.5, we obtain: $\beta_0(\vec{P}(\mathbb{R}^2_1)^{(k,l)}_{\mathbf{0}}) = \binom{k+l}{k}$.

1.8. **Outline of the paper.** In Section 2 we construct, for an arbitrary Euclidean cubical complex K, a homotopy equivalence between $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ and a certain homotopy colimit of spaces which are homotopy equivalent to one of the "smaller" spaces $\vec{P}(K)^{\mathbf{l}}_{\mathbf{0}}$ for $\mathbf{l} < \mathbf{k}$. In Section 3, we construct a homomorphism $A_*(K) \to H_*(\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}})$. Then we prove, under the assumption that $[\mathbf{0},\mathbf{k}]_{(n-1)} \subseteq K$, that this homomorphism is actually an isomorphism. The proof is inductive and uses the homotopy colimit description from Section 2. In Section 4, we determine the cohomology ring structure $H^*(\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}})$.

2. A recursive description of path spaces

In this Section we construct a presentation of directed paths spaces on a Eulidean cubical complex as a homotopy colimit of path spaces of certain subcomplexes. Fix $\mathbf{0} \ll \mathbf{k} \in \mathbb{Z}^n$ and a cubical complex $K \subseteq [\mathbf{0},\mathbf{k}]$.

2.1. **A transversal section.** Fix $\varepsilon \in (0,1)$.
Let $S(K) = \{\mathbf{x} \in K :\ |\mathbf{x}| = |\mathbf{k}| - \varepsilon\} \subset S = \{\mathbf{x} \in \mathbb{R}^n |\ |\mathbf{x}| = |\mathbf{k}| - \varepsilon\}$.

Proposition 2.1. *For every path $\alpha \in \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ there exists a unique $s(\alpha) \in S(K)$ belonging to the image of α. Moreover, the map $s : \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \to S(K)$ is continuous (with respect to the compact-open topology on $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$).*

Proof. Since $|\alpha(0)| = 0$, $|\alpha(1)| = |\mathbf{k}|$, there exists t_α such that $\alpha(t_\alpha) = |\mathbf{k}| - \varepsilon$. If $|\alpha(t_\alpha)| = |\alpha(t'_\alpha)|$ for $t_\alpha < t'_\alpha$, then $\alpha(t_\alpha) \leq \alpha(t'_\alpha)$. This implies that $|\alpha(t_\alpha) - \alpha(t'_\alpha)| = ||\alpha(t_\alpha)| - |\alpha(t'_\alpha)|| = 0$. Therefore $s(\alpha)$ is uniquely determined. In the parlance of Raussen [9], the subset $S(X)$ is both achronal and unavoidable from $\mathbf{0}$ to $\mathbf{k}$.

To prove the continuity of s, it is sufficient to show that the inverse images of open balls $B(\mathbf{x},r) = \{\mathbf{y} \in K :\ |\mathbf{x} - \mathbf{y}| < r\}$ are open in $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$: Fix $\mathbf{x} \in K$, $r > 0$, $\alpha \in s^{-1}(B(\mathbf{x},r))$ and $t_\alpha \in I$ such that $\alpha(t_\alpha) = s(\alpha)$. Let $\beta \in \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ be a path such that $|\beta(t_\alpha) - s(\alpha)| < r'$, where $r' = (r - |\mathbf{x} - s(\alpha)|)/2$. Since $s(\beta)$ and $\beta(t_\alpha)$ are comparable, we have

$$|\beta(t_\alpha) - s(\beta)| = ||\beta(t_\alpha)| - |s(\beta)|| = ||\beta(t_\alpha)| - |s(\alpha)|| < r'.$$

Finally,

$$|\mathbf{x} - s(\beta)| \leq |\mathbf{x} - s(\alpha)| + |s(\alpha) - \beta(t_\alpha)| + |\beta(t_\alpha) - s(\beta)| < |\mathbf{x} - s(\alpha)| + 2r' < r,$$

i. e. $\beta \in s^{-1}(B(\mathbf{x},r))$. Hence the set $\{\beta \in \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} :\ \beta(t_\alpha) \in B(s(\alpha),r')\}$ is an open neighbourhood of α contained in $s^{-1}(B(\mathbf{x}),r)$. □

2.2. **A description of $S(K)$.** The map

$$R : S \ni \mathbf{t} \mapsto \varepsilon^{-1}(\mathbf{k} - \mathbf{t}) \in \{\mathbf{x} :\ \mathbf{x} \geq 0\ \wedge\ |\mathbf{x}| = 1\} = |\Delta^{n-1}|$$

is a homeomorphism from S to the standard simplex Δ^{n-1}. It maps $S(K)$ homeomorphically onto a simplicial subcomplex $\Delta_K \subset \Delta^{n-1}$.

The category Δ^{op}_{n-1} of subsimplices of Δ_{n-1} can be identified by an isomorphism of categories with the (inverse) poset category $\mathcal{J}_{n-1}$ of sequences $\mathbf{j} \in \{0,1\}^n$ with $\mathbf{j} \neq \mathbf{0}$. Such a sequence $\mathbf{j}$ corresponds to the subsimplex

$$\Delta_{\mathbf{j}} = \{\mathbf{t} \in \Delta_{n-1} : \ \forall_{i=1}^n \quad j_i = 0 \Rightarrow t_i = 0\} \subseteq \Delta_{n-1}.$$

The morphism $\mathbf{j} \to \mathbf{j}'$ (for every $\mathbf{j} \geq \mathbf{j}'$) corresponds to the inclusion $\Delta_{\mathbf{j}} \subset \Delta_{\mathbf{j}'}$.

The restriction of this correspondence to the category of subsimplices of Δ_K provides an isomorphism between that category and the full subcategory $\mathcal{J}_K \subset \mathcal{J}_{n-1}$ with objects

$$\text{Ob}(\mathcal{J}_K) := \{\mathbf{0} < \mathbf{j} \in \{0,1\}^n : \ \mathbf{j} \subseteq \Delta_K\} = \{\mathbf{j} \in \{0,1\}^n : \ [\mathbf{k}-\mathbf{j}, \mathbf{k}] \subseteq K\}. \tag{2.2}$$

Two cases will be of particular importance:

$$\mathcal{J}_K = \begin{cases} \mathcal{J}_{n-1} & [\mathbf{k}-\mathbf{1}, \mathbf{k}] \subset K; \\ \hat{\mathcal{J}}_{n-1} := \mathcal{J}_{n-1} \setminus \{\mathbf{1}\} & [\mathbf{k}-\mathbf{1}, \mathbf{k}] \cap K = \partial[\mathbf{k}-\mathbf{1}, \mathbf{k}]. \end{cases}$$

2.3. **A cover of $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$.** The geometric realization of $S(K)$ can be covered by stars of its vertices and this cover lifts to a cover of $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$. For every $\mathbf{j} \in \mathcal{J}_K$ define

$$F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} := (R \circ s)^{-1}(\text{st}(\mathbf{j})) = \{\alpha \in \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} : \ \forall_{i=1}^n \ j_i = 1 \Rightarrow s(\alpha)_i < k_i\}.$$

The spaces $F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ clearly cover all of $\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$. The cover $\{F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}\}$ is closed under intersections since $F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \cap F_{\mathbf{j}'}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} = F_{\mathbf{j}\cup\mathbf{j}'}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$, where $(\mathbf{j} \cup \mathbf{j}')_i = \max\{j_i, j'_i\}$. Moreover the category associated with this cover is precisely $\mathcal{J}_K$. As a consequence, cf Segal [14, Proposition 4.1], the inclusions $F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \subseteq \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ induce a homotopy equivalence

$$\text{hocolim}_{\mathbf{j}\in\mathcal{J}_K} F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \longrightarrow \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} = \text{colim}_{\mathbf{j}\in\mathcal{J}_K} F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}. \tag{2.3}$$

The next step is to prove that $F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$ is homotopy equivalent to $\vec{P}(K)^{\mathbf{k}-\mathbf{j}}_{\mathbf{0}}$. This will enable us to use the decomposition (2.3) for inductive calculations of path spaces. But first, we need some technical lemmas which will be presented in a greater generality.

2.4. **Past deformation retractions.**

Definition 2.4. Let X be a d-space with a subspace $Y \subseteq X$. *A past deformation retraction* of X onto Y is a d-map $d : X \times \vec{I} \to X$ (preserving d-structures; cf Grandis [7]) such that

- $r(x) := d(x, 0) \in Y$ for every $x \in X$,
- $d(x, 1) = x$ for every $x \in X$,
- $d(y, t) = y$ for every $y \in Y$ and every $t \in I$.

Proposition 2.5. *If $d : X \times \vec{I} \to X$ is a past deformation retraction on Y, then for every $x \in X$ and $y \in Y$ the maps*

$$F : \vec{P}(Y)^{d(x,0)}_y \ni \alpha \mapsto \alpha * d(x, -) \in \vec{P}(X)^x_y$$
$$G : \vec{P}(X)^x_y \ni \alpha \mapsto d(\alpha, 0) \in \vec{P}(Y)^{d(x,0)}_y$$

are mutually inverse homotopy equivalences.

Proof. A homotopy H between $Id_{\vec{P}(Y)_y^{d(x,0)}}$ and $G \circ F$ is given by the formula

$$H(\alpha, s)(t) = \begin{cases} \alpha(t(1-s/2)^{-1}) & \text{for } 0 \leq t \leq 1-s/2 \\ d(x,0) & \text{for } 1-s/2 \leq t \leq 1, \end{cases}$$

and a homotopy H' between $Id_{\vec{P}(X)_y^x}$ and $F \circ G$ by

$$H'(\beta, s)(t) = \begin{cases} d(\beta(t(1-s/2)^{-1}), 1-s) & \text{for } 0 \leq t \leq 1-s/2 \\ d(x, 2t-1) & \text{for } 1-s/2 \leq t \leq 1. \end{cases} \qquad \square$$

2.5. $F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ **up to homotopy.** For $\mathbf{j} \in \mathcal{J}_K$ define

$$X_{\mathbf{j}} := \{\mathbf{t} \in K : \forall_{i:\mathbf{j}_i=1}\, t_i < k_i\} \tag{2.6}$$

$$K_{\mathbf{j}} := \{\mathbf{t} \in K : \mathbf{t} \leq \mathbf{k} - \mathbf{j}\} = K \cap [\mathbf{0}, \mathbf{k} - \mathbf{j}], \tag{2.7}$$

and let $\bar{X}_{\mathbf{j}}$ be the closure of $X_{\mathbf{j}}$ in K.

Proposition 2.8. *$K_{\mathbf{j}}$ is a past deformation retract of $\bar{X}_{\mathbf{j}}$.*

Proof. Every $\mathbf{t} \in \bar{X}_{\mathbf{j}}$ belongs to some cube $[\mathbf{c}, \mathbf{d}] \subseteq K$, $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ whose interior is contained in $X_{\mathbf{j}}$. It implies that $c_i = k_i - 1$ for every i such that $d_i = k_i$ and $j_i = 1$. Define the retraction $r^{\mathbf{j}} : \bar{X}_{\mathbf{j}} \to K_{\mathbf{j}}$ by the formula

$$r_i^{\mathbf{j}}(\mathbf{t}) = \begin{cases} t_i & \text{for } t_i \leq k_i - j_i \\ k_i - j_i & \text{for } t_i \geq k_i - j_i, \end{cases}$$

and the deformation between $r^{\mathbf{j}}$ and identity by convex combination. Since both $\mathbf{t}$ and $r^{\mathbf{j}}(\mathbf{j})$ belong to the cube $[\mathbf{c}, \mathbf{d}]$ the map $r^{\mathbf{j}}$ is well-defined. $\square$

Proposition 2.9. *The image $\alpha(I)$ associated to any path $\alpha \in F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ is contained in $\bar{K}_{\mathbf{j}}$.*

Proof. Let $\alpha \in F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ and let $t_\alpha \in I$ satisfies $\alpha(t_\alpha) = s(\alpha)$. For $t \leq t_\alpha$, we have $\alpha(t)_i \leq \alpha(t_\alpha)_i < k_i$ whenever $\mathbf{j}_i = 1$. Then $\alpha([0, t_\alpha]) \subseteq K_{\mathbf{j}} \subseteq \bar{K}_{\mathbf{j}}$. If $t > t_\alpha$, then $\alpha(t) \in [\mathbf{k} - \mathbf{j}', \mathbf{k}]$, where $[\mathbf{k} - \mathbf{j}', \mathbf{k}]$ is a minimal cube containing $s(\alpha)$ in its interior. As a consequence, $\alpha([t_\alpha, 1]) \subseteq \bar{K}_{\mathbf{j}}$ and hence $F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \subseteq \vec{P}(\bar{K}_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}}$. $\square$

For every cube $[\mathbf{c}, \mathbf{d}] \subseteq K$ let $i_{\mathbf{c}}^{\mathbf{d}} : \vec{P}(K)_{\mathbf{0}}^{\mathbf{c}} \to \vec{P}(K)_{\mathbf{0}}^{\mathbf{d}}$ denote the concatenation with the linear path $t \mapsto (1-t)\mathbf{c} + t\mathbf{d}$. Note that $i_{\mathbf{k}-\mathbf{j}}^{\mathbf{k}}(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}) \subseteq F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ for $\mathbf{j} \in \mathcal{J}_K$.

Proposition 2.10. *For every $\mathbf{j} \in \mathcal{J}_K$ the map $i_{\mathbf{k}-\mathbf{j}}^{\mathbf{k}} : \vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} \to F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ is a homotopy equivalence. Moreover, for every morphism $\mathbf{j}' \to \mathbf{j}$ in $\mathcal{J}_K$ the diagram*

$$\begin{tikzcd} \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}'} \arrow[r, "i_{\mathbf{k}-\mathbf{j}'}^{\mathbf{k}-\mathbf{j}}"] \arrow[d, "i_{\mathbf{k}-\mathbf{j}'}^{\mathbf{k}}"] & \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} \arrow[d, "i_{\mathbf{k}-\mathbf{j}}^{\mathbf{k}}"] \\ F_{\mathbf{j}'}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \arrow[r, "\subseteq"] & F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \end{tikzcd} \tag{2.11}$$

commutes up to homotopy.

Proof. The map $i_{\mathbf{k}-\mathbf{j}}^{\mathbf{k}}$ is a homotopy equivalence by Proposition 2.5. The commutativity of the diagram is obvious from the definitions. $\square$

2.6. **Specific path spaces.**

2.6.1. *Boundary of a cube.* Let $\mathbf{k} = \mathbf{1}$ and let $K = [\mathbf{0}, \mathbf{1}]_{(n-1)}$. Then $S(K) \simeq \partial\Delta^{n-1}$ and $\mathcal{J}_K = \mathcal{J}_{n-1} \setminus \{\mathbf{1}\}$ with objects $\{\mathbf{j} \in \{0,1\}^n : \ 0 < \sum_{i=1}^n j_i < n\}$. Furthermore, for every such $\mathbf{j}$, $F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{1}} \simeq \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} = \vec{P}[\mathbf{0}, \mathbf{k}-\mathbf{j}]_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}$ is contractible since it contains $\{\mathbf{0}\}$ as a past deformation retract, cf Proposition 2.5. As a consequence,

$$\vec{P}(\mathbb{R}^n_{(n-1)})_{\mathbf{0}}^{\mathbf{1}} \cong \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}(K)} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} \simeq N\mathcal{J}(K) \cong \partial\Delta^{n-1} \simeq S^{n-2}.$$

Remark. This result is also an immediate consequence of Raussen [11, Corollary 4.12].

For the remaining part of the paper, we fix a generator $x_{\mathbf{1}} \in H_{n-2}(\vec{P}([\mathbf{0},\mathbf{1}]_{(n-1)})_{\mathbf{0}}^{\mathbf{1}}) \cong H_{n-2}(|N\mathcal{J}|)$. By shifting, we obtain the generators $x_{\mathbf{k}} \in H_{n-2}(\vec{P}([\mathbf{k}-\mathbf{1},\mathbf{k}]_{(n-1)})_{\mathbf{k}-\mathbf{1}}^{\mathbf{k}})$ for $\mathbf{k} \in \mathbb{Z}^n$.

2.6.2. *Connectivity of certain path spaces.*

Proposition 2.12. *If a subcomplex $K \subseteq [\mathbf{0}, \mathbf{k}]$ contains the 2-skeleton of $[0, \mathbf{k}]$, then $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ is connected.*

Proof. This is obvious if either $\mathbf{k} = \mathbf{0}$, or $n = 2$. Assume that the conclusion holds for all complexes $K' \subseteq [\mathbf{0}, \mathbf{k}']$, $\mathbf{k}' \in \mathbb{Z}^{n'}$, such that $n' \leq n$ or $n' = n$ and $\mathbf{k}' < \mathbf{k}$.
For $\mathbf{k} > \mathbf{0}$, $S(K) \subseteq \Delta^{n-1}$ contains the 1-skeleton of Δ^{n-1} and is therefore connected. Then

$$\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}_K} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$$

is connected, because it is a homotopy colimit of connected spaces $F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} = \vec{P}(K \cap [\mathbf{0}, \mathbf{k}-\mathbf{j}])_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}$ (by the inductive assumption) over a connected category $\mathcal{J}_K$. □

3. Homology of the path space $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$

Fix $n \geq 3$, $\mathbf{k} \in \mathbb{Z}^n$. Let $K \subseteq [\mathbf{0}, \mathbf{k}] \subseteq \mathbb{R}^n$ denote a Euclidean cubical complex which contains the $(n-1)$-skeleton of $[\mathbf{0}, \mathbf{k}]$. We will define a homomorphism $\Phi_K : A_*(K) \to H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$ from the graded abelian group $A_*(K)$ defined in Section 1.4 into the homology of the path space and prove that it is an isomorphism.

3.1. **The homomorphism Φ_K.**

3.1.1. *Definitions.* For every cube sequence $\mathbf{a}^* = [\mathbf{0} \ll \mathbf{a}^1 \ll \cdots \ll \mathbf{a}^r] \in CS_r(K)$ in K choose paths $\beta_i \in \vec{P}(K)_{\mathbf{a}^i}^{\mathbf{a}^{i+1}-\mathbf{1}}$, $i = 0, \dots, r$ (we assume $\mathbf{a}^0 = \mathbf{0}$, $\mathbf{a}^{r+1} = \mathbf{k}+\mathbf{1}$). Let $c(\mathbf{a}^*)$ be the following concatenation map

$$c(\mathbf{a}^*) : \prod_{j=1}^{r} \vec{P}(K)_{\mathbf{a}^j-\mathbf{1}}^{\mathbf{a}^j} \ni (\alpha_j) \mapsto \beta_0 * \alpha_1 * \beta_1 * \cdots * \alpha_r * \beta_r \in \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}.$$

Then define Φ_K on generators by

$$\Phi_K([\mathbf{a}^1 \ll \cdots \ll \mathbf{a}^r]) := c(\mathbf{a}^*)_*(x_{\mathbf{a}^1} \times x_{\mathbf{a}^2} \times \cdots \times x_{\mathbf{a}^r}) \in H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}), \tag{3.1}$$

and extend as a homomorphism. The element $\mathbf{x}_{\mathbf{a}^i} \in H_{n-2}(\vec{P}(K)_{\mathbf{a}^j-\mathbf{1}}^{\mathbf{a}^j}) \cong H_{n-2}(\vec{P}(\partial[\mathbf{a}^j-\mathbf{1}, \mathbf{a}^j])_{\mathbf{a}^j-\mathbf{1}}^{\mathbf{a}^j})$ is a generator chosen as in Section 2.6.1. According to Proposition 2.12, the map $c(\mathbf{a}^*)$ does not depend (up to homotopy) on the choice of the paths β_i, and this implies that Φ_K is well-defined.

3.1.2. *Naturality.* Let $\mathbf{l} \leq \mathbf{k}$ and let $L \subseteq [\mathbf{0}, \mathbf{l}]$ be a cubical complex such that $[\mathbf{0}, \mathbf{l}]_{(n-1)} \subseteq L \subseteq K \cap [\mathbf{0}, \mathbf{l}]$. With respect to the homomorphism given by

$$\varphi_L^K : A_*(L) \ni [\mathbf{a}^*] \mapsto \begin{cases} [\mathbf{a}^*] & \text{if } [\mathbf{a}^*] \text{ is a cube sequence in } K \\ 0 & \text{otherwise.} \end{cases} \in A_*(K),$$

the homomorphism Φ_K is natural in the following sense:

Proposition 3.2. *The diagram*

$$\begin{tikzcd} A_*(L) \arrow[r, "\varphi_L^K"] \arrow[d, "\Phi_L"] & A_*(K) \arrow[d, "\Phi_K"] \\ H_*(\vec{P}(L)_{\mathbf{0}}^{\mathbf{l}}) \arrow[r] & H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \end{tikzcd} \tag{3.3}$$

is commutative. The bottom map is induced by the concatenation with a fixed directed path $\alpha \in \vec{P}(K)_{\mathbf{l}}^{\mathbf{k}}$.

Proof. Straightforward from the definitions. □

3.2. **The main theorem.** The main result of this section is the following

Theorem 3.4. *For every Euclidean cubical complex $[\mathbf{0}, \mathbf{k}]_{(n-1)} \subseteq K \subseteq [\mathbf{0}, \mathbf{k}]$, $\mathbf{k} \in \mathbb{Z}^n$, $\mathbf{k} \geq \mathbf{0}, n > 2$, the homomorphism $\Phi_K : A_*(K) \to H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$ is an isomorphism of graded abelian groups.*

The proof is by induction on $\mathbf{k}$. To start the induction, notice that if $\prod_i k_i = 0$, then both $H_*(K)_{\mathbf{0}}^{\mathbf{k}}$ and $A_*(K)$ are isomorphic to $(\mathbb{Z}, 0)$, the homology of a point, since $[\mathbf{0}, \mathbf{k}]_{(n-1)} \subseteq K$. Let us assume that Theorem 3.4 is valid for all Euclidean cubical complexes contained in $[\mathbf{0}, \mathbf{l}]$ for $\mathbf{l} < \mathbf{k}$.

Since K is assumed to contain the $(n-1)$-skeleton of $[\mathbf{0}, \mathbf{k}]$, there are only two cases to consider: either $[\mathbf{k}-\mathbf{1}, \mathbf{k}]$ is contained in K, or it is not; in that case $[\mathbf{k}-\mathbf{1}, \mathbf{k}] \cap K = \partial[\mathbf{k}-\mathbf{1}, \mathbf{k}]$. For simplicity, we will write $\mathcal{J} = \mathcal{J}_{n-1}$, resp. $\hat{\mathcal{J}} = \hat{\mathcal{J}}_{n-1}$ for the relevant categories; cf Section 2.2. Let $\mathbf{Ab}_*$ be the category of graded abelian groups.

3.2.1. *The case $[\mathbf{k}-\mathbf{1}, \mathbf{k}] \subseteq K$.* The objects of the category $\mathcal{J}$ are all n-tuples $\mathbf{0} < \mathbf{j} \in \{0, 1\}^n$, cf (2.2). For $\mathbf{j} \in \{0, 1\}^n$ denote $K_{\mathbf{j}} := K \cap [\mathbf{0}, \mathbf{k} - \mathbf{j}]$. Notice that for any morphism $\mathbf{j} \to \mathbf{j}'$ in $\mathcal{J}$, the homomorphisms

$$\begin{aligned} \varphi_{K_{\mathbf{j}}}^{K_{\mathbf{j}'}} &: A_*(K_{\mathbf{j}}) \to A_*(K_{\mathbf{j}'}) \\ (i_{\mathbf{k}-\mathbf{j}}^{\mathbf{k}-\mathbf{j}'})_* &: H_*(\vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}) \to H_*(\vec{P}(K_{\mathbf{j}'})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}'}) \\ (incl)_* &: H_*(F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})) \to H_*(F_{\mathbf{j}'}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \end{aligned}$$

define functors $A_*(K_{(-)}), H_*(\vec{P}(K_{(-)})_{\mathbf{0}}^{\mathbf{k}-(-)}, F_{(-)}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} : \mathcal{J} \to \mathbf{Ab}_*$; compare Proposition 3.2 and Proposition 2.10.

Proposition 3.5. *If* $[\mathbf{k}-\mathbf{1},\mathbf{k}]\subseteq K$*, the homomorphism* Φ_K *is the composition*

$$A_*(K) \overset{\cong}{\leftarrow} \mathrm{colim}_{\mathbf{j}\in\mathcal{J}} A_*(K_{\mathbf{j}}) \xrightarrow{\mathrm{colim}_{\mathbf{j}}\Phi_{K_{\mathbf{j}}}} \mathrm{colim}_{\mathbf{j}\in\mathcal{J}} H_*(\vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}) \xrightarrow{\mathrm{colim}_{\mathbf{j}}(i^{\mathbf{k}}_{\mathbf{k}-\mathbf{j}})_*}$$
$$\mathrm{colim}_{\mathbf{j}\in\mathcal{J}} H_*(F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \xrightarrow{Q} H_*(\mathrm{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \overset{\cong}{\to} H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}).$$

with Q *the colimit of the maps* $Q_{\mathbf{j}}: H_*(F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \to H_*(\mathrm{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$*,* $\mathbf{j}\in\mathcal{J}$*. Moreover, all these homomorphisms are isomorphisms.*

Proof. It is easy to check that the homomorphism $\mathrm{colim}_{\mathbf{j}\in\mathcal{J}} A_*(K_{\mathbf{j}}) \to A_*(K)$ induced by inclusions $\varphi^K_{K_{\mathbf{j}}}$ is an isomorphism – since $[\mathbf{k}-\mathbf{1},\mathbf{k}]\subseteq K$. Thus, $A_*(K)$ is generated by cube sequences $[\mathbf{a}^*]$ in $K_{\mathbf{j}}$. Now the conclusion follows from Proposition 3.2 applied for pairs $K_{\mathbf{j}}\subseteq K$. Furthermore, $\mathrm{colim}_{\mathbf{j}}\Phi_{K_{\mathbf{j}}}$ is an isomorphism by the induction hypothesis, $\mathrm{colim}_{\mathbf{j}}(i^{\mathbf{k}}_{\mathbf{k}-\mathbf{j}})_*$ by Proposition 2.10 and for the last isomorphism by (2.3). We are left to show that also Q is an isomorphism; this will be achieved in Proposition 3.8. □

Proposition 3.6. *The compositions*

$$A_*(K_{\mathbf{j}}) \xrightarrow{\Phi_{K_{\mathbf{j}}}} H_*(\vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}) \xrightarrow{(i^{\mathbf{k}}_{\mathbf{k}-\mathbf{j}})_*} H_*(F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$$

define a natural equivalence of functors $A_*(K_{(-)})$ *and* $H_*(F_{(-)}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$ *from* $\mathcal{J}$ *into the category of graded abelian groups.*

Proof. Both homomorphisms are isomorphisms by the inductive hypothesis and Proposition 2.10. The naturality of the transformations is a consequence of Propositions 3.2 and 2.10. □

Proposition 3.7. *For every* $t\geq 0$*, the functor* $A_t(K_{(-)})$ *is a projective object in the category of functors* $\mathcal{J}\to\mathbf{Ab}$.

Proof. If $n-2$ does not divide t, then by definition $A_t(K_{(-)})=0$ is projective. Assume that $t=(n-2)q$, $q\in\mathbb{Z}$. We find a presentation of $A_t(K_{(-)})$ as a direct sum of projective summands: Within the set $CS_q(K)$ of cube sequences in K (cf Section 1.4) let

$$X_{\mathbf{j}} = CS_q(K_{\mathbf{j}}) \setminus \bigcup_{\mathbf{j}<\mathbf{h}\in\mathcal{J}} CS_q(K_{\mathbf{h}}).$$

Next, define functors $M_{\mathbf{j}}:\mathcal{J}\to\mathbf{Ab}$ by

$$M_{\mathbf{j}}(\mathbf{h}) = \begin{cases}\mathbb{Z}[X_{\mathbf{j}}] & \text{if } \mathbf{h}\leq\mathbf{j}\\ 0 & \text{otherwise,}\end{cases};$$

the morphisms are identities whenever possible, and trivial otherwise. Immediately from the definitions we obtain that $CS_q(K_{\mathbf{j}})=\bigcup_{\mathbf{h}\geq\mathbf{j}} X_{\mathbf{h}}$ and hence $A_t(K_{(-)})\cong\bigoplus_{\mathbf{j}\in\mathcal{J}} M_{\mathbf{j}}$.

For an arbitrary functor $N:\mathcal{J}\to\mathbf{Ab}$ the set of transformations $\mathrm{Hom}_{\mathcal{J}}(M_{\mathbf{j}},N)$ is naturally isomorphic to $\mathrm{Hom}(\mathbb{Z}[X_{\mathbf{j}}],N(\mathbf{j}))$. Therefore the projectivity of $\mathbb{Z}[X_{\mathbf{j}}]$ implies that the functors $M_{\mathbf{j}}$ are also projective. As a consequence, $A_t(K_{(-)})\cong\bigoplus_{\mathbf{j}\in\mathcal{J}} M_{\mathbf{j}}$ is projective. □

Proposition 3.8. *The homomorphism*

$$Q:\mathrm{colim}_{\mathbf{j}\in\mathcal{J}} H_*(F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \longrightarrow H_*(\mathrm{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$$

is an isomorphism.

Proof. Following Bousfield and Kan [2, XII.5.7], there is a spectral sequence

$$E^2_{s,t} = \operatorname{colim}^s_{\mathcal{J}_k} H_t(F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}) \Rightarrow H_*(\operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}),$$

where colim^s stands for s-th left derived functor of colim. In fact, $E^2_{s,t} = 0$ for $s > 0$ since

$$H_t(F_{(-)}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}) \simeq A_t(K_{(-)})$$

is projective (Proposition 3.6 and 3.7). Hence the spectral sequence degenerates to the isomorphism $\operatorname{colim}_{\mathbf{j}\in\mathcal{J}} H_*(F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}) \cong H_*(\operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}})$. □

Corollary 3.9. Assume that $[\mathbf{k}-\mathbf{1},\mathbf{k}] \subseteq K$ and $\Phi_{K_{\mathbf{j}}}$ is an isomorphism for $\mathbf{j} \in \mathcal{J}$. Then Φ_K is an isomorphism, as well.

Proof. This follows immediately from Proposition 3.5 and 3.8. □

3.2.2. *The case* $[\mathbf{k}-\mathbf{1},\mathbf{k}] \not\subseteq K$. Denote $L := K \cup [\mathbf{k}-\mathbf{1},\mathbf{k}]$ and denote $\hat{\mathcal{J}} := \hat{\mathcal{J}}_{n-1}$, and $\mathcal{J} := \mathcal{J}_{n-1}$ with objects

$$\operatorname{Ob}(\hat{\mathcal{J}}) = \{\mathbf{j} \in \{0,1\}^n : \ \mathbf{0} < \mathbf{j} < \mathbf{1}\} \subseteq \{\mathbf{j} \in \{0,1\}^n : \ \mathbf{0} < \mathbf{j}\} = \operatorname{Ob}(\mathcal{J}).$$

Their nerves $N\hat{\mathcal{J}}, N\mathcal{J}$ have geometric realizations $S^{n-2} \cong \partial\Delta^{n-1} \cong |N\hat{\mathcal{J}}| \subset |N\mathcal{J}| \cong \Delta^{n-1}$. Consider the sequence of cofibrations

$$\begin{tikzcd}
\vec{P}(K_{\mathbf{1}})^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \times |N\hat{\mathcal{J}}| \arrow[r] \arrow[d, equal] & \vec{P}(K_{\mathbf{1}})^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \times |N\mathcal{J}| \arrow[d, equal] \\
\vec{P}(L)^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \times |N\hat{\mathcal{J}}| \arrow[r] \arrow[d, "\simeq"'] & \vec{P}(L)^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \times |N\mathcal{J}| \arrow[d, "\simeq"'] \\
\operatorname{hocolim}_{\mathbf{j}\in\hat{\mathcal{J}}} \vec{P}(L)^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \arrow[r] \arrow[d, "\simeq"', "i^{\mathbf{k}}_{\mathbf{k}-\mathbf{1}}"] & \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} \vec{P}(L)^{\mathbf{k}-\mathbf{1}}_{\mathbf{0}} \arrow[d, "\simeq"', "i^{\mathbf{k}}_{\mathbf{k}-\mathbf{1}}"] \\
\operatorname{hocolim}_{\mathbf{j}\in\hat{\mathcal{J}}} F_{\mathbf{1}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \arrow[r] \arrow[d, hook] & \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{1}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \arrow[d, hook] \\
\operatorname{hocolim}_{\mathbf{j}\in\hat{\mathcal{J}}} F_{\mathbf{j}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \arrow[r] & \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \arrow[d, equal] \\
\operatorname{hocolim}_{\mathbf{j}\in\hat{\mathcal{J}}} F_{\mathbf{j}}\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \arrow[r] \arrow[u, hook, "\simeq"', "2.10"] \arrow[d, "\simeq"', "2.3"] & \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}} F_{\mathbf{j}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \arrow[d, hook, "\simeq"', "2.3"] \\
\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}} \arrow[r] & \vec{P}(L)^{\mathbf{k}}_{\mathbf{0}}
\end{tikzcd} \tag{3.10}$$

In all the squares of the diagram apart from the middle one the vertical maps are homotopy equivalences; hence the cofibres are also homotopy equivalent. One can easily check, using the construction of the homotopy colimit, that the maps in the middle square induce a homeomorphism between cofibres of type $F_{\mathbf{1}}\vec{P}(L)^{\mathbf{k}}_{\mathbf{0}} \times S^{n-1}$. The diagram above induces a transfomation between associated homology long exact sequences. In particular, the following diagram

$$\begin{tikzcd}
H_{*+1}(\vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \times |N\mathcal{J}|, \vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \times |N\hat{\mathcal{J}}|) \arrow[r, "1\times\partial_{\mathcal{J}}"] \arrow[d, "\mu"', "\cong"] & H_*(\vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \times |N\hat{\mathcal{J}}|) \arrow[d, "\nu"'] \\
H_{*+1}(\vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}, \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \arrow[r, "\partial_P"] & H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})
\end{tikzcd} \tag{3.11}$$

is commutative, where

$$\partial_{\mathcal{J}} : H_{*+1}(|N\mathcal{J}|, |N\hat{\mathcal{J}}|) \to H_*(|N\hat{\mathcal{J}}|)$$

is the differential of the long homology exact sequence of the pair $(|N\mathcal{J}|, |N\hat{\mathcal{J}}|)$ and where μ is an isomorphism. Consider the composition

$$\begin{aligned}\Psi : A_{*-(n-2)}(K_{\mathbf{1}}) \xrightarrow{\Phi_{K_{\mathbf{1}}}} {} & H_{*-(n-2)}\vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \xrightarrow{\times \partial_{\mathcal{J}}^{-1}(x_{\mathbf{k}})} \\ & H_{*+1}(\vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \times |N\mathcal{J}'|, \vec{P}(K_{\mathbf{1}})_{\mathbf{0}}^{\mathbf{k-1}} \times |N\mathcal{J}|) \xrightarrow{\mu} H_{*+1}(\vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}, \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}),\end{aligned}$$

where $x_{\mathbf{k}} \in H_{n-2}(|N\hat{\mathcal{J}}|)$ is a fixed generator. By the inductive assumption, $\Phi_{K_{\mathbf{1}}}$ is an isomorphism and so is Ψ. Define

$$\varrho : A_{*-(n-2)}(K_{\mathbf{1}}) \ni [\mathbf{a}^1 \ll \dots \ll \mathbf{a}^r] \mapsto [\mathbf{a}^1 \ll \dots \ll \mathbf{a}^r \ll \mathbf{k}] \in A_*(K), \tag{3.12}$$

Proposition 3.13. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & A_{*-(n-2)}(K_{\mathbf{1}}) \arrow[r, "\varrho"] \arrow[d, "\Psi"] & A_*(K) \arrow[r, "\varphi_K^L"] \arrow[d, "\Phi_K"] & A_*(L) \arrow[r] \arrow[d, "\Phi_L"] & 0 \\
0 \arrow[r] & H_{*+1}(\vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}, \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \arrow[r, "\partial_P"] & H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) \arrow[r, "(K\subseteq L)_*"] & H_*(\vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}) \arrow[r] & 0
\end{tikzcd}$$

is commutative, and it has exact rows. Moreover, all vertical homomorphisms are isomorphisms.

Proof. Exactness of the upper row follows immediately from definitions. The right square commutes by Proposition 3.2. Since Φ_L is an isomorphism (by Proposition 3.5), the composition $\Phi_L \circ \varphi_K^L$ is surjective. Hence $(K \subseteq L)_*$ is also surjective and this implies exactness of the lower row. For every cube sequence $[\mathbf{a}^*]$ in $K_{\mathbf{1}}$, we have

$$\begin{aligned}\partial_P(\Psi([\mathbf{a}^*])) = \partial_P(\mu(\Phi_{K_{\mathbf{1}}}([\mathbf{a}^*]) \times \partial_{\mathcal{J}}^{-1}(x_{\mathbf{k}}))) = \nu(\Phi_{K_{\mathbf{1}}}([\mathbf{a}^*]) \times x_{\mathbf{k}}) = \\ = \nu(\Phi_{K_{\mathbf{1}}}([\mathbf{a}^*]) \times \Phi_{[\mathbf{k-1},\mathbf{k}]_{(n-1)}}([\mathbf{k}])) = \Phi_K([\mathbf{a}^* \ll \mathbf{k}]) = \Phi_K(\varrho([\mathbf{a}^*])).\end{aligned}$$

Hence the left square commutes. Finally, since both Ψ and Φ_L are isomorphisms, Φ_K is an isomorphism by the Five Lemma. □

Proof of Theorem 3.4. Induction starts for $\mathbf{k} \not\gg \mathbf{0}$ as stated immediately after the statement of Theorem 3.4. Assume that the theorem holds for all $\mathbf{l} < \mathbf{k}$. By Corollary 3.9, Φ_K is an isomorphism whenever $[\mathbf{k-1}, \mathbf{k}] \subseteq K$ and by Proposition 3.13, it holds for any $K \subseteq [\mathbf{0}, \mathbf{k}]$. □

3.3. **A generalization.** The main Theorem 3.4 applies of course also to spaces that are homotopy equivalent to the path spaces $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$. To obtain such spaces consider a functor $Z : \mathbb{Z}_+^n \to \mathbf{Top}$ from the poset of non-negative n-dimensional vectors (regarded as a category) obeying to

$$Z(\mathbf{k}) \begin{cases} \text{contractible}, & \prod_i k_i = 0 \\ \simeq \operatorname{hocolim}_{j \in \mathcal{C}_{\mathbf{k}}} Z(\mathbf{k} - \mathbf{j}), \mathcal{C}_{\mathbf{k}} = \begin{cases} \mathcal{J}, & [\mathbf{k}-\mathbf{1}, \mathbf{k}] \subset K \\ \hat{\mathcal{J}}, & [\mathbf{k}-\mathbf{1}, \mathbf{k}] \not\subset K \end{cases}, & \prod_i k_i \neq 0. \end{cases} \tag{3.14}$$

A particular simple such functor functor Z_0 can be constructed recursively by

$$Z_0(\mathbf{k}) = \begin{cases} * \text{ (a one point space)} & \prod_i k_i = 0 \\ \operatorname{hocolim}_{j \in \mathcal{C}_{\mathbf{k}}} Z(\mathbf{k} - \mathbf{j}) & \prod_i k_i \neq 0 \end{cases}$$

with $\mathcal{C}_{\mathbf{k}}$ as above.

Proposition 3.15. *Functors $Z_i, Z_j : \mathbb{Z}_+^n \to \mathbf{Top}$ obeying to (3.14) yield homotopy equivalent spaces $Z_i(\mathbf{k}) \simeq Z_j(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{Z}_+^n$.*

Proof. This can be seen inductively (for $Z_j = Z_0$) starting from constant maps $Z_j(\mathbf{k}) \to Z_0(\mathbf{k})$ for $\prod_i k_i = 0$ extending to

$$Z_i(\mathbf{k}) \simeq \operatorname{hocolim}_{j \in \mathcal{C}_{\mathbf{k}}} Z_i(\mathbf{k} - \mathbf{j}) \simeq \operatorname{hocolim}_{j \in \mathcal{C}_{\mathbf{k}}} Z_0(\mathbf{k} - \mathbf{j}) \simeq Z_o(\mathbf{k}). \qquad \square$$

In particular, the functor $Z(\mathbf{k}) = \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ obeys to (3.14), and hence $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq Z_0(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{Z}_+^n$. We shall now present a "sub"functor Z_1 also obeying to (3.14) that can serve to motivate the main Theorem 3.4:

Cube sequences between $\mathbf{0}$ and $\mathbf{k}$ are partially ordered by inclusion. A cube sequence is *maximal* if $\mathbf{a}^i \not\ll \mathbf{a}^{i+1} - \mathbf{1}$ for all i. Let $\bar{C}S(K)_{\mathbf{0}}^{\mathbf{k}}$ denote the set of maximal cube sequences between $\mathbf{0}$ and $\mathbf{k}$. For a maximal cube sequence $\mathbf{a}^*$, let $P(\mathbf{a}^*) = \prod_{j=1}^r \vec{P}(K)_{\mathbf{a}^j - \mathbf{1}}^{\mathbf{a}^j} \times \vec{P}(K)_{\mathbf{a}^j}^{\mathbf{a}^{j+1} - \mathbf{1}}$; by maximaity, the latter factor is always contractible, and hence $P(\mathbf{a}^*) \simeq \prod_{j=1}^r \vec{P}(K)_{\mathbf{a}^j - \mathbf{1}}^{\mathbf{a}^j} \simeq \prod_{j=1}^r (S^{n-2})^r$.

Concatenation defines maps $\bar{c}(\mathbf{a}^*) : P(\mathbf{a}^*) \to \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ assembling to $\bar{c} : \coprod_{\mathbf{a}^* \in \bar{C}S(K)_{\mathbf{0}}^{\mathbf{k}}} \vec{P}(\mathbf{a}^*) \to \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$. The image defines a subspace $\vec{P}'(K)_{\mathbf{0}}^{\mathbf{k}} \subset \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ of d-paths through integral points that – alternatingly – have at least one coordinate in common or for which *every* coordinate is the successor of the previous one.

Corollary 3.16. Inclusion $\vec{P}'(K)_{\mathbf{0}}^{\mathbf{k}} \subset \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ is a homotopy equivalence for all $\mathbf{0} \leq \mathbf{k}$.

Proof. It is easy to check that the functor $Z_1(\mathbf{k}) = \vec{P}'(K)_{\mathbf{0}}^{\mathbf{k}}$ obeys to (3.14). Apply Proposition 3.15. $\square$

4. The cohomology ring of the path space $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$

Fix $\mathbf{k} \in \mathbb{Z}^n$ and a Euclidean cubical complex $K \subseteq [\mathbf{0}, \mathbf{k}]$ containing its $(n-1)$-skeleton. As proven in the previous Section 3, the homology of the path space $\vec{P}(K)_{\mathbf{k}}^{\mathbf{0}}$ is isomorphic, as a graded group, to $A_*(K)$. Since this group is free, the cohomology of $\vec{P}(K)_{\mathbf{k}}^{\mathbf{0}}$ is isomorphic to its dual, i.e. there is a sequence of isomorphisms

$$\Phi^K : H^*(\vec{P}(K)_{\mathbf{k}}^{\mathbf{0}}) \cong \operatorname{Hom}(H_*(\vec{P}(K)_{\mathbf{k}}^{\mathbf{0}}), \mathbb{Z}) \xrightarrow{\operatorname{Hom}(\Phi_K, \mathbb{Z})} A^*(K) := \operatorname{Hom}(A_*(K), \mathbb{Z}). \tag{4.1}$$

Let $Z^*(K)$ denote the free graded exterior $\mathbb{Z}$-algebra with generators the cube sequences $[\mathbf{0} \ll \mathbf{l} \leq \mathbf{k}], [\mathbf{l}-\mathbf{1}, \mathbf{l}] \not\subset K$ of length 1. Let $I(K)$ denote the ideal generated by products $\mathbf{l}_1\mathbf{l}_2$ with $\mathbf{l}_1 \not\ll \mathbf{l}_2$ and $\mathbf{l}_2 \not\ll \mathbf{l}_1$. Let $F^*(K)$ denote the quotient algebra $F^*(K) = Z^*(K)/I^*(K)$, a free abelian group with the cube sequences $[\mathbf{a}^*]$ in K as basis. Moreover, we can provide $A^*(K)$ with a ring structure via the $\mathbb{Z}$-module isomorphism $\Psi^K : A^*(K) \to F^*(K)$, $\phi \mapsto \sum_{\mathbf{a}^* \in CS(K)} \phi(\mathbf{a}^*)\mathbf{a}^*$.

Proposition 4.2. *The map* $\Psi^K \circ \Phi^K : H^*(\vec{P}(K)^{\mathbf{0}}_{\mathbf{k}}) \to F^*(K)$ *is a graded ring isomorphism.*

Proof. Fix a cube sequence $\mathbf{a}^*$ in K, giving rise to inclusion (by concatenation, as in (3.1)) $c(\mathbf{a}^*) : \prod_{i=1}^r \vec{P}(K)^{\mathbf{a}^i}_{\mathbf{a}^i-\mathbf{1}} \to \vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}$. To $\mathbf{a}^*$ corresponds

- a graded abelian group $i(\mathbf{a}^*) : A_*(\mathbf{a}^*) \subset A_*(K)$ generated by the set of *sub*-cube sequences of $\mathbf{a}^*$ with $i(\mathbf{a}^*)$ the inclusion homomorphism
- a free graded exterior algebra $F^*(\mathbf{a}^*) = Z^*(\mathbf{a}^*)$ with generators $\mathbf{a}^i$ and a projection homomorphism $p(\mathbf{a}^*) : F^*(K) \to F^*(\mathbf{a}^*)$ and
- an additive isomorphism $\Psi(\mathbf{a}^*) : A^*(\mathbf{a}^*) \to F^*(\mathbf{a}^*)$.

Moreover, the group isomorphism $\Phi(\mathbf{a}^*) : A_*(\mathbf{a}^*) \to H_*(\prod_{i=1}^r \vec{P}(K)^{\mathbf{a}^i}_{\mathbf{a}^i-\mathbf{1}})$ has a dual $\Phi(\mathbf{a}^*)^* : H^*(\prod_{i=1}^r \vec{P}(K)^{\mathbf{a}^i}_{\mathbf{a}^i-\mathbf{1}}) \to A^*(\mathbf{a}^*)$ fitting into the commutative diagram

(4.3)
$$\begin{tikzcd}
H^*(\vec{P}(K)^{\mathbf{k}}_{\mathbf{0}}) \arrow[r, "c(\mathbf{a}^*)"] \arrow[d, "\Phi^K"] & H^*(\prod_{i=1}^r \vec{P}(K)^{\mathbf{a}^i}_{\mathbf{a}^i-\mathbf{1}}) \arrow[d, "\Phi(\mathbf{a}^*)^*"] \\
A^*(K) \arrow[r, "i(\mathbf{a}^*)"] \arrow[d, "\Psi^K"] & A^*(\mathbf{a}^*) \arrow[d, "\Psi(\mathbf{a}^*)"] \\
F^*(K) \arrow[r, "p(\mathbf{a}^*)"] & F^*(\mathbf{a}^*)
\end{tikzcd}$$

All vertical maps are isomorphisms of abelian groups, and all maps, apart from possibly Φ^K, are ring homomorphisms.

The assembly map $\bigoplus_{\mathbf{a}^* \in CS(K)} A_*(\mathbf{a}) \to A_*(K)$ is clearly surjective whence its dual $\bigoplus_{\mathbf{a}^* \in CS(K)} i(\mathbf{a}^*) : \bigoplus_{\mathbf{a}^* \in CS(K)} A^*(K) \to \bigoplus_{\mathbf{a}^* \in CS(K)} A^*(\mathbf{a}^*)$ injects. Hence, Φ^K is a ring isomorphism, as well.

□

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Department of Mathematical Sciences, Aalborg University, Fredrik Bajersvej 7G, DK-9220 Aalborg Øst, Denmark

E-mail address: raussen@math.aau.dk

URL: http://people.math.aau.dk/~raussen/

Faculty of Mathematics, Informatics and Mechanics, Warsaw University, Banacha 2, PL-02-097 Warszawa, Poland

E-mail address: ziemians@mimuw.edu.pl

URL: http://www.mimuw.edu.pl/~ziemians/